Developing a concrete-pictorial-abstract model for negative number arithmetic (2016)

Redhill Teaching School Alliance

NOTTINGHAM TRENT UNIVERSITY

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Context

Negative number arithmetic is frequently cited by both students and teachers as being challenging to learn, and challenging to teach. Vlassis (2008) and Bofferding (2010) found that the multiple roles of the negative sign, as a *unary*, *binary*, and *symmetric* operator, presents a fundamental challenge for learners. A fourth conceptual challenge arises from the fact that while the magnitude of positive numbers can be visualised in terms of the cardinality of sets, negative numbers cannot. The challenge for teachers lies in effectively addressing these four conceptual elements in the instructional model. According to Askew and Wiliam (1995), students will "constantly 'invent' rules to explain the patterns they see around them." This may explain why negative number arithmetic is a topic so fraught with misconceptions; if the instructional approach does not address conceptual understanding, students will actively seek out rules and justifications which may not necessarily be conceptually sound. In 2012 the UK Department for Education found that a common feature of successful mathematics curricula is a greater emphasis on conceptual understanding (DfE, 2012). The *concrete-pictorial-abstract* (CPA) sequence is an instructional approach practised in Singapore since the 1980s, and has recently gained popularity in the UK as an approach to promote conceptual understanding. It involves development of conceptual understanding by first exploring a concept through the use of concrete manipulatives; next representing that concept pictorially; and finally representing the concept symbolically. In the context of negative number arithmetic in UK secondary schools, this approach differs from conventional practice in its use of the concrete and pictorial stages. Witzel (2005) showed that this approach was effective in teaching algebra to middle-school students in the US, and Altiparmak and Ozdogan (2008) found that achievement and progress were significantly better for sixth-grade students in Turkey who were introduced to negative number concepts using a visual, constructivist approach than for students taught using a traditional approach.

Further reading

- Bofferding, L., 2010, Addition and subtraction with negatives: Acknowledging the multiple meanings of the minus sign.
- Bruner, J., 1966, Toward a Theory of Instruction.
- Teppo, M., Heuvel-Panhuizen, M., 2013, Visual representations as objects of analysis: the number line as an example.
- Vlassis, J., 2008, The Role of Mathematical Symbols in the Development of Number Conceptualization: The Case of the Minus Sign.
- Witzel, B., 2005, Using CRA to Teach Algebra to Students with Math Difficulties in Inclusive Settings.
- Yew Hoong, L., Weng Kin, H., Lu Pien, C., 2015, Concrete-Pictorial-Abstract: Surveying its origins and charting its future.

Aim of this study

This study aims to develop a CPA model for negative number arithmetic, using the number line and 'number bars' as the key representations. The design attempts to effectively address the four conceptualisations of unary, binary, and symmetric operations, and magnitude.

Methodology

The participating students (n = 12) were aged between 14 and 15 years, with the trial group (n = 7) and control group (n = 5) forming two existing classes which were both ranked fifth out of five sets in the year group, based on prior attainment.

Score	Maximum possible score
Overall score	14
Ordering numbers	2
Addition	4
Subtraction	4
Multiplication	2
Division	2

The trial group was taught using the trial CPA model, and the control group was taught using a conventional non-CPA approach. The trial unit was a sequence of 8 lessons, amounting to 8 hours in total over the academic year. The conventional approach was to revisit the topic three separate times during the year, amounting to 10 hours in total over the academic year.

Research questions

Is there a difference in change in score from pre-assessment to post-assessment between trial and control group?

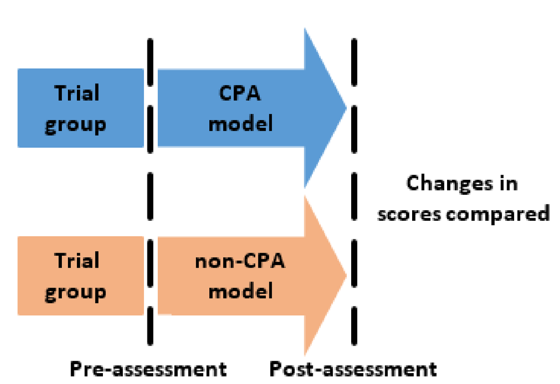
Is there qualitative evidence that trial or control group demonstrate conceptual understanding?

Quantitative evidence

Both trial and control groups completed a pre-assessment prior to studying the topic of negative number arithmetic and a post-assessment afterwards. The changes in students' scores from pre- to post-assessment were compared in six areas:

Qualitative evidence

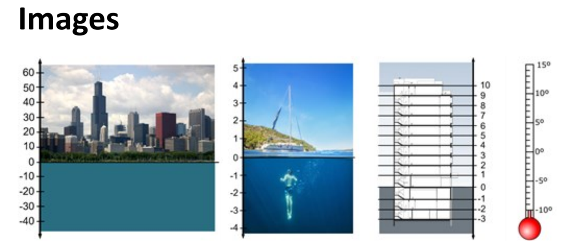
Lessons were observed by an in-class observer; lessons were filmed and observed after the unit had been delivered; and post-lesson discussions were held with both trial and control group class teachers.



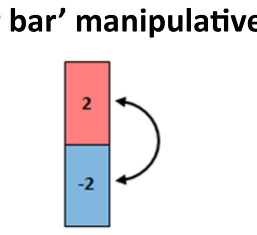
Design of the CPA model for negative number arithmetic

Stage	Aspect	Conceptualisation	Representation	Rationale
Concrete		Unary, symmetric, binary, magnitude	Thermometer; Sea level; Building section with basement levels	Position, direction, and movement may be represented in concrete contexts.
Pictorial		Unary, symmetric, binary, magnitude	Vertical number line	Direct abstraction from the concrete representations.
Pictorial		Symmetric, binary, magnitude	Number bar manipulatives	Sign as direction and magnitude as distance from zero.
Abstract	Vocabulary	Unary, magnitude	Higher Lower Zero	To avoid possible confusion over the terms *smaller*; *larger*; *greater*; *less*. Zero as a position on the number line as opposed to a zero-quantity. Negative numbers 'count downwards', away from zero. Misconceptions about magnitude, for example, a student might think that (– 7) > 4 since 7 > 4.
Abstract	Vocabulary	Symmetric, magnitude	Negative Opposite Direction Up Down	The terms *negative* and *opposite* are equivalent. The sign of a number denotes the direction in which it is pointing. Avoiding use of the term *minus*, which could be confused with the subtraction operation.
Abstract	Vocabulary	Addition (binary)	Addition Forward	Addition as forward movement.
Abstract	Vocabulary	Subtraction (binary)	Subtraction Backward	Subtraction as backward movement.
Abstract	Vocabulary	Multiplication (binary)	Multiplication Add on … times. Lots of	Multiplication as repeated addition.
Abstract	Vocabulary	Division (binary)	Division Take away ... times. How many times does … go into …?	Division as repeated subtraction.
Abstract	Notation	Unary, symmetric	Use of brackets around negative numbers.	To distinguish between the negative sign of a number and the subtraction operation.
Abstract	Formal rules	Addition (unary, symmetric, binary)	Adding a negative number is equivalent to subtracting a positive number.	Procedural fluency based on conceptual understanding.
Abstract	Formal rules	Subtraction (unary, symmetric, binary)	Subtracting a negative number is equivalent to adding a negative number.	
Abstract	Formal rules	Multiplication (unary, symmetric, binary)	The product of an even/odd number of negative numbers is positive/negative.	
Abstract	Formal rules	Division (unary, symmetric, binary)	The quotient of an even/odd number of negative numbers is positive/negative.	

Images

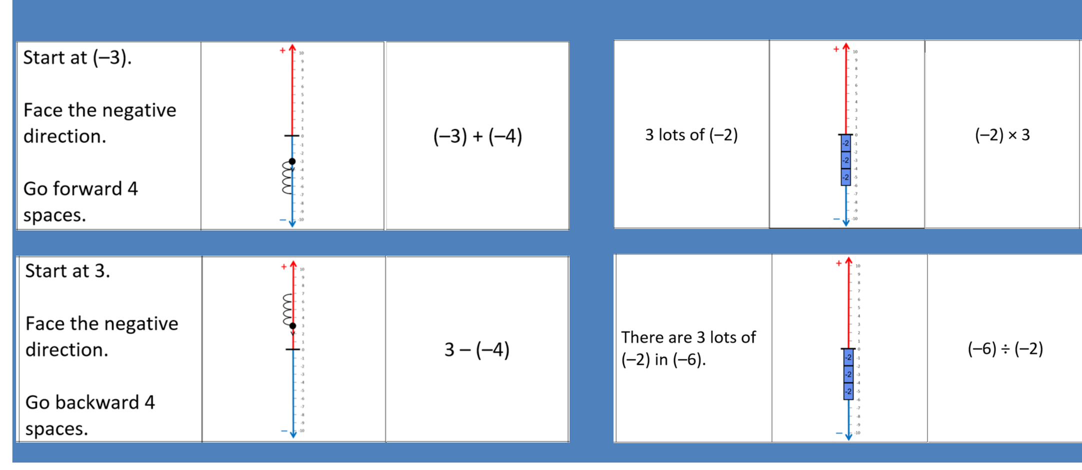


'Number bar' manipulatives



Addition and subtraction using the number line

Multiplication and division using the number line



Results

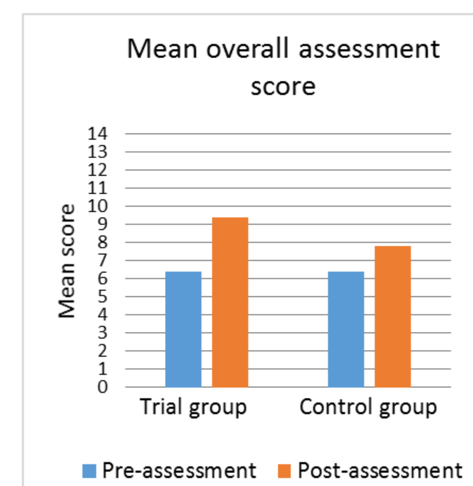
A comparison of the changes in scores from pre-assessment to post-assessment in the areas of ordering numbers and division reveals little or no difference between the trial and control groups.

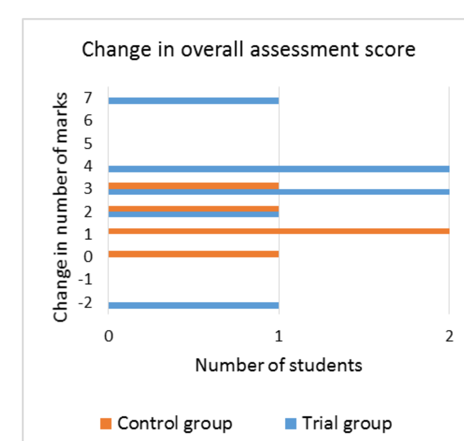
Overall score

The mean increase for the trial group was 3 marks, compared with 1.4 marks for the control group. This agrees with the median increases which were 3 marks and 1 mark for trial and control groups respectively.

There was greater variation in the score increase in the trial group (ranging from –2 to 7), compared to the control group (ranging from 0 to 3).

A t-test (two-tailed) suggests that this difference in mean increase is significant (p = 0.025 < 0.05), suggesting that the majority of students in the trial group increased their scores by a significantly greater number of marks than the majority of students in the control group.



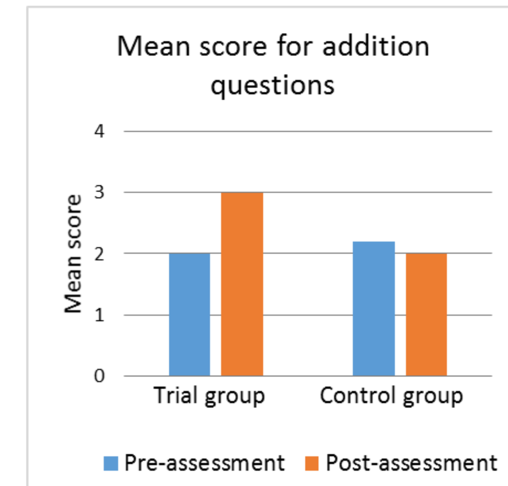


Addition

The mean increase for the trial group was 1 mark, compared with –0.2 marks for the control group. This agrees with the median increases which were 1 mark and 0 marks for trial and control groups respectively.

There was greater variation in the number of marks gained in the trial group (ranging from –1 to 3), compared to the control group (ranging from –1 to 0).

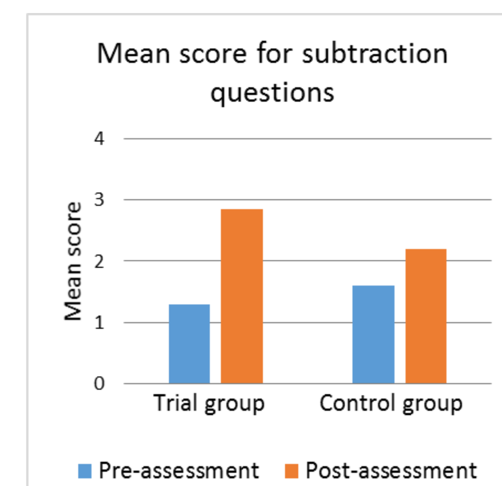
A t-test (two-tailed) suggests that this difference in mean increase is significant (p = 0.034 < 0.05), suggesting that the majority of students in the trial group increased their scores by a significantly greater number of marks than the majority of students in the control group.



Subtraction

The mean increase for the trial group was 1.8 marks, compared with a mean increase of 0.6 marks in the control group. This agrees less closely with the median increases which were 2 marks and 0 marks for trial and control groups respectively.

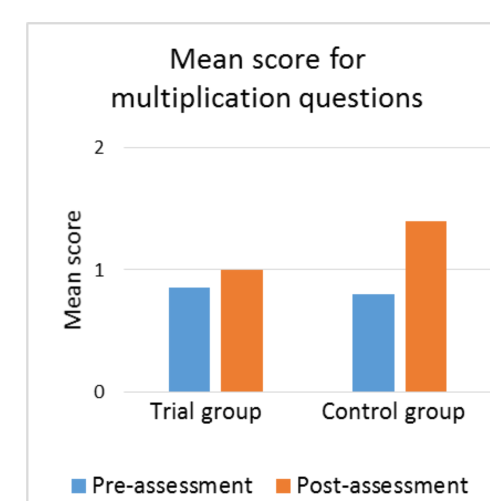
A t-test (two-tailed) suggests that this difference in mean increase is significant (p = 0.040 < 0.05), suggesting that the majority of students in the trial group increased their scores by a significantly greater number of marks than the majority of students in the control group.



Multiplication

The mean increase for the trial group was 0.17 marks, compared with 0.75 marks for the control group. This agrees with the median increases which were 0 marks and 1 mark for trial and control groups respectively.

A t-test (two-tailed) suggests that this difference in mean increase is not significant (p = 0.058 > 0.05), suggesting that the majority of students in the control group did not increase their scores by a significantly greater number of marks than the majority of students in the trial group.



Lesson observations and post-lesson discussions

The class teacher and in-class observer both noted that there was evidence of increased student independence when using number lines to add and subtract. Since students were able to see that their answers were correct, they did not need the teacher to verify their answers. The class teacher also noted that this level of independence was uncharacteristic of this class.

There was evidence from classwork that students were initially using the number line to answer questions and then progressing on to answer more challenging questions without use of the number lines. This suggests that some internalisation of concept may have been taking place.

The solutions written by two students from the trial group in the pre-assessment and the addition lesson are shown below:

	Pre-assessment	Addition lesson task
Student A	(–4) + 2 = –6 7 + (-1) = –8	(–1) + (–4) + 2 = –3 2 + (–3) + 4 + (–3) = 0
Student B	(–5) + (–3) = –2	(–4) + (–1) + 2 + (–5) + (–3) = –11

The class teacher noted that the vocabulary of 'negative as opposite' was effective in increasing students' confidence and understanding of the concept of double-negative. The double-negative was not included in the pre- and post-assessments so it is unclear how students progressed with this concept.

The consistent use of vocabulary and notation became an unexpected motivating factor in lessons, with students enjoying the opportunity to correct each others' use of the word 'minus' instead of 'negative', and to suggest where brackets should be written in order to make expressions easier to understand.

Conclusion

The results of this study suggest that the majority of students who were taught negative number concepts using a CPA approach increased their scores in post-assessments by a significantly greater number of marks than students who were taught using a non-CPA approach in the areas of addition and subtraction with negative numbers. There is also evidence to suggest that the CPA model had a positive effect on student confidence, independence, and engagement.

In conclusion, the CPA model using the vertical number line and 'number bars' as key representations had a positive effect on student progress in addition and subtraction with negative numbers, but would require further development in order to effectively represent multiplication and division.

Discussion

This was a small-scale project with a broad scope; the intention was not to provide rigorous evidence, or to promote a particular approach, but to see if there were any interesting outcomes when we trialled a unit based on the CPA approach. The use of a very small sample and short assessments means that it is difficult to determine whether or not the results are statistically significant. When viewed as a whole, however, in conjunction with the qualitative evidence, and the fact that the control group had received a total of 10 hours of lessons on negative numbers by the time they completed the post-assessment, compared with 8 hours for the trial group, the results suggest that the CPA approach had a positive effect on student progress, confidence, and independence. There is also evidence to suggest that the CPA approach may have allowed students to access the concepts more easily through concrete and pictorial representations, and hence develop greater conceptual understanding. This study raises the question of whether a CPA approach leads to greater conceptual understanding in learners, compared to a non-CPA approach. This may be a suitable question to investigate further, perhaps trough a longitudinal study of concept-skill retention, or by investigating how effectively learners are able to apply negative number concepts to solve problems in unfamiliar situations.

Next steps

Appropriate next steps towards developing this model further would be to trial the model with high-attaining students. This learning unit was designed to include the formal rules of negative number arithmetic; however, in the judgement of the class teacher, the students in this sample were unable to move past the pictorial stage. It could be argued that this provides some further evidence to support the assertion that the number line model has allowed the students to access and progress with a topic that would have otherwise been less accessible.

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